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Multidimensional matrix characterizations of the Banach and Pringsheim core

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Abstract

In this paper we shall present a multidimensional invariant Pringsheim core theorem. Conditions on a four-dimensional matrix transformation that will ensure that the transformed Pringsheim core of a bounded double sequence $[x]$ is contained in the double Banach core of $[x]$ shall also be presented.
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1. Introduction

In 1930 Knopp introduced the concept of the core of a complex sequence. Following Knopp's work Patterson presented a multidimensional definition for the core of a double sequence which is as follows; let $P - C_n\{x\}$ to be the least closed convex set that includes all points $x_{k,l}$ for $k, l > n$; then the Pringsheim core of the double sequence $[x]$ is the set $P-C\{x\} = \bigcap_{n=1}^{\infty} P-C_n\{x\}$. Using this definition Patterson also presented the following analogue to the Knopp core theorem; if A is a nonnegative four-dimensional regular summability matrix, then $P-C\{Ax\} \subseteq P-C\{x\}$ for any bounded sequence $[x]$ for which $[Ax]$ exists. In this paper we shall present conditions on four-dimensional matrices that leave the Pringsheim core invariant; this will be accomplished by establishing a multidimensional analogue of the following theorem: In 2000 Yardimci presented the following theorem: Let $A = [a_{n,k}]$ be a real matrix and $x = [x_k]$ a real bounded sequence. Then $\mathcal{K} - \text{core}(Ax) = \beta - \text{core}(x)$ if and only if

(1) $A = [a_{n,k}]$ is strongly regular,
(2) $\lim_n \sum_{k=1}^{\infty} |a_{n,k}| = 1$, and
(3) for every infinite sequence of suffixes p_i $(i = 1, 2, 3, \ldots)$, the number 1 is a limit point of the sequence $u_n = \sum_i a_{n,p_i}$.

In addition, we shall establish a theorem of inclusion between the Banach core and the Pringsheim core.

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2. Definitions, notation and preliminary results

Definition 2.1 (*Pringsheim, [10]*). A double sequence $x = [x_{k,l}]$ has **Pringsheim limit** L (denoted by $P\text{-}\lim x = L$) provided that given $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > N$. We shall describe such an x more briefly as **"P-convergent"**.

Definition 2.2 (*Pringsheim, [10]*). A double sequence $[x]$ is called **definite divergent** if for every (arbitrarily large) $G > 0$ there exist two natural numbers n_1 and n_2 such that $|x_{n,k}| > G$ for $n \geq n_1, k \geq n_2$.

Definition 2.3 (*Patterson, [7]*). The double sequence $[y]$ is a double **subsequence** of the sequence $[x]$ provided that there exist two increasing double-index sequences $\{n_j\}$ and $\{k_j\}$ such that if $z_j = x_{n_j,k_j}$, then y is formed by

$$\begin{array}{cccc} z_1 & z_2 & z_5 & z_{10} \\ z_4 & z_3 & z_6 & - \\ z_9 & z_8 & z_7 & - \\ - & - & - & -. \end{array}$$

Definition 2.4 (*Patterson, [7]*). A number β is called a **Pringsheim limit point** of the double sequence $[x]$ provided that there exists a subsequence $[y]$ of $[x]$ that has Pringsheim limit $\beta : P\text{-}\lim[y] = \beta$.

Silverman and Toeplitz in [12] and [13], respectively, presented the notion of regularity for two-dimensional matrix transformations. The definition is as follows: a four-dimensional matrix transformation is said to be regular if it maps every convergent sequence into a convergent sequence with the same limit. Following this work Robison in 1926 presented a four-dimensional analogue of regularity for double sequences in which he added an additional assumption of boundedness. This assumption was made because a double sequence which is P-convergent is not necessarily bounded. Along these same lines, Robison and Hamilton presented a Silverman–Toeplitz type of multidimensional characterization of regularity in [1] and [11]. The definition of the regularity for four-dimensional matrices will be stated next, followed by the Robison–Hamilton characterization of the regularity of four-dimensional matrices.

Definition 2.5. The four-dimensional matrix A is said to be **RH-regular** if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.

Theorem 2.1. *The four-dimensional matrix A is RH-regular if and only if*

RH_1: $P\text{-}\lim_{m,n} a_{m,n,k,l} = 0$ *for each k and l;*
RH_2: $P\text{-}\lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} = 1$;
RH_3: $P\text{-}\lim_{m,n} \sum_{k=1}^{\infty} |a_{m,n,k,l}| = 0$ *for each l;*
RH_4: $P\text{-}\lim_{m,n} \sum_{l=1}^{\infty} |a_{m,n,k,l}| = 0$ *for each k;*
RH_5: $\sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}|$ *is P-convergent; and*
RH_6: *there exist positive numbers A and B such that*

$$\sum_{k,l>B} |a_{m,n,k,l}| < A.$$

In [2] Knopp introduced the concept of the core of a complex number sequence. Following that idea Patterson presented the following definition of the core of a double sequence in [8].

Definition 2.6. Let $P - C_n\{x\}$ be the least closed convex set that includes all points $x_{k,l}$ for $k, l > n$; then the **Pringsheim core** of the double sequence $[x]$ is the set $P - C\{x\} = \cap_{n=1}^{\infty} P - C_n\{x\}$.

Using this definition Patterson also presented the following multidimensional analogue of the Knopp Core theorem in [8].

Theorem 2.2. *If A is a nonnegative RH-regular summability matrix, then $P - C\{Ax\} \subseteq P - C\{x\}$ for any bounded sequence $[x]$ for which $[Ax]$ exists.*

Now let us state the definition for absolutely equivalent RH-regular matrices.

Definition 2.7. Two RH-regular matrices A and B are said to be **absolutely equivalent** for a given class of sequence $[x_{k,l}]$ whenever

$$P\text{-}\lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} (a_{m,n,k,l} - b_{m,n,k,l}) x_{k,l} = 0.$$

In [9] the following proposition, lemma, and theorem characterizing the relationship between absolutely equivalent matrices and the Pringsheim core appeared.

Proposition 2.1. *A necessary and sufficient condition for the RH-regular matrices $A = [a_{m,n,k;l}]$ and $B = [b_{m,n,k;l}]$ to be absolutely equivalent for all bounded sequences is that*

$$P\text{-}\lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l} - b_{m,n,k,l}| = 0.$$

Lemma 2.1. *If two double sequences $[s_{k,l}]$ and $[s'_{k,l}]$ are such that*

$$P\text{-}\lim_{k,l} |s_{k,l} - s'_{k,l}| = 0$$

then $P - C\{s\} = P - C\{s'\}$.

Theorem 2.3. *The Pringsheim core of a transformation $A = [a_{m,n,k,l}]$ on a bounded sequence $[s_{k,l}]$ is contained in the core of $[s_{k,l}]$ if and only if A is RH-regular and is absolutely equivalent to a nonnegative matrix $B = [b_{m,n,k,l}]$ for all bounded sequences.*

The concept of almost convergent sequences (ordinary — single dimensional) was introduced by Lorentz in [3]. Recently Móricz and Rhoades introduced the following notion of almost P-convergent sequences.

Definition 2.8. A double sequence $x = [x_{k,l}]$ of real numbers is said to be **almost P-convergent** to a limit s if

$$P\text{-}\lim_{p,q} \sup_{m,n\geq 0} \left| \frac{1}{pq} \sum_{k,l=m,n}^{m+p-1,n+q-1} x_{k,l} - s \right| = 0,$$

that is, the average value of $[x]$ taken over any rectangle $\{(k,l) : m \leq k \leq m+p-1 \cap n \leq l \leq n+q-1\}$ tends to s as both p and q tend to infinity in the Pringsheim sense, and this is uniform in m and n.

If a double sequence $[x]$ is almost P-convergent to s then we write $f_2 - \lim x = s$ and denote the set of almost P-convergent sequences by f_2. Note that a convergent single sequence is also almost convergent; however, this is not the case for double sequences (i.e. a P-convergent double sequence need not be almost P-convergent). However, every bounded P-convergent sequence is also almost P-convergent. In [4] Móricz and Rhoades presented a notion of strongly RH-regular matrices as follows:

Theorem 2.4. *Necessary and sufficient conditions for a matrix $A = [a_{m,n,k;l}]$ to be strongly RH-regular are that A is RH-regular and satisfies the following two conditions:*

(1) $P\text{-}\lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} |\triangle_{10} a_{m,n,k,l}| = 0$ *and*
(2) $P\text{-}\lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} |\triangle_{01} a_{m,n,k,l}| = 0$

where $\triangle_{10} a_{m,n,k,l} = a_{m,n,k,l} - a_{m,n,k+1,l}$ and $\triangle_{01} a_{m,n,k,l} = a_{m,n,k,l} - a_{m,n,k,l+1}$ for $k,l = 0,1,2,3,\ldots$.

Mursaleen and Edely in [5] considered the following transformation

$$L_2^*(x) = \limsup_{p,q} \sup_{m,n\geq 0} \frac{1}{pq} \sum_{k,l=m,n}^{m+p-1,n+q-1} x_{k,l}$$

and defined the double Banach core of a real-valued bounded double sequence $[x]$ to be the closed interval $[-L_2^*(-x), L_2^*(x)]$ (denoted $\beta^2 - C(x)$). Since every bounded P-convergent sequence is also almost P-convergent it follows that $L_2^*(x) \leq P\text{-}\limsup x$. Thus $\beta^2 - C(x) \subseteq P - C(x)$ for all bounded double sequences. With this concept Mursaleen and Edely presented the following theorem:

Theorem 2.5. *For every bounded double sequence* $[x]$, $P\text{-}\limsup Ax \leq L_2^*(x)$ *(i.e.* $P - C(Ax) \subseteq \beta^2 - C(x)$*) if and only if*

(1) $A = [a_{m,n,k;l}]$ *is strongly RH-regular and*
(2) $P\text{-}\lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}| = 1$.

In 2000 Yardimci presented the following theorem in [14]:

Theorem 2.6. *Let* $A = [a_{n,k}]$ *be a real matrix and* $x = [x_k]$ *be a real bounded sequence. Then* $\mathcal{K} - \text{core}(x) = \beta^2 - \text{core}(x)$ *if and only if*

(1) $A = [a_{n,k}]$ *is strongly regular,*
(2) $\lim_n \sum_{k=1}^{\infty} |a_{n,k}| = 1$, *and*
(3) *for every infinite sequence of suffixes* p_i $(i = 1, 2, 3, \ldots)$*, the number 1 is a limit point of the sequence* $u_n = \sum_i a_{n,p_i}$.

In 2004 Orhan and Yardimci presented the following theorem in [6]:

Theorem 2.7. *Let* $[x]$ *be a bounded sequence and let A be a strongly regular matrix. Then* $\mathcal{K}-\text{core}(Ax) \subseteq \beta-\text{core}(x)$ *if and only if A is absolutely equivalent to a nonnegative strongly regular matrix B for all bounded sequences.*

In the next section we shall present multidimensional analogues of Theorems 2.6 and 2.7.

3. Main results

Theorem 3.1. *Let* $A = [a_{m,n,k;l}]$ *be a four-dimensional real-valued matrix and* $[x]$ *a real bounded double sequence. Then*

$$P - C\{Ax\} = \beta^2 - C\{x\}$$

if and only if

(1) *A is strongly RH-regular,*
(2) $P\text{-}\lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}| = 1$, *and*
(3) *for every infinite double sequence of suffices* (p_i, q_j); $i, j = 1, 2, 3, \ldots$, *the following holds*

$$P\text{-}\lim_{m,n} \sum_{i,j=1,1}^{\infty,\infty} a_{m,n,p_i,q_j} = 1.$$

Proof. Let $P - C\{Ax\} \subseteq \beta^2 - C\{x\}$ (i.e. $P\text{-}\limsup Ax \leq L_2^*(x)$). Then Theorem 3.1 of [5] grants us (1) and (2) holds. Now to establish part (3), let us consider the following double sequence:

$$x_{k,l} = \begin{cases} 1, & \text{if } k = p_i \text{ and } l = q_j;\ i, j = 1, 2, 3, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

If there exists M such that for $k, l > M$, $x_{k,l} \neq 0$, then $P\text{-}\lim x = 1$ and since $[x]$ is a bounded P-convergent sequence then $[x]$ is almost P-convergent. This implies $\beta^2 - C(x) = \{1\}$. However $P - C(Ax) \subseteq \beta^2 - C(x) = \{1\}$. Thus $P\text{-}\lim Ax = 1$. Hence

$$P\text{-}\lim_{m,n} \sum_{i,j} a_{m,n,p_i,q_j} = 1.$$

Now for the case when $\beta^2 - C(x) \subseteq P - C(x)$, let us consider a P-divergent sequence consisting of only 0's and 1's; then $P - C(x) = [0, 1]$. Therefore there exist two subsequences as in Definition 2.3, say $[u]$ and $[v]$ such that $P\text{-}\lim u = 1$ and $P\text{-}\lim v = 0$. Also it is clear that $f_2 - \lim u = 1$ and $f_2 - \lim v = 0$. Thus

$$[0, 1] \subseteq \beta^2 - C(x) \subseteq P - C(x) \subseteq [0, 1].$$

By assumption $P - C(Ax) = \beta^2 - C(x) = [0, 1]$. Thus (3) is necessary.

Suppose (1) through (3) hold. Then Theorem 3.1 of [5] grants us $P - C\{Ax\} \subseteq \beta^2 - C\{x\}$ for all bounded double sequences. Thus we need only to show that $\beta^2 - C\{x\} \subseteq P - C\{Ax\}$. To achieve this goal we shall show that the set of f_2-limit points of a double sequence $[x]$ is a subset of the set of P-limit points of $[Ax]$. Let $[x]$ be a bounded double sequence and let λ be a f_2-limit point of $[x]$. Thus there exist p_i and q_j such that $f_2 - \lim_{i,j} x_{p_i,q_j} = \lambda$. Therefore x_{p_i,q_j} can be expressed as $x_{p_i,q_j} = \lambda + \epsilon_{p_i,q_j}$ where $f_2 - \lim_{i,j} \epsilon_{p_i,q_j} = 0$. Let

$$\begin{aligned}
&D = \{(p_i, q_j) : i, j = 1, 2, 3, \ldots\},\\
&D' = (\mathbb{N} \times \mathbb{N})\backslash D,\\
&D_1 = \{(p, q) : 1 \le p < \infty \cap 1 \le q < \bar{N}\},
\end{aligned}$$

and

$$D_2 = \{(p, q) : \bar{N} \le p < \bar{M} \cap \bar{N} \le q < \infty\}.$$

Observe that (3) implies that $P\text{-}\lim_{\alpha,\beta} \sum_{(p,q)\in D} a_{m_\alpha,n_\beta,p,q} = 1$. Let us consider the following transformation

$$\begin{aligned}
y_{m,n} = {} & \sum_{(p,q)\in D} a_{m,n,p,q} x_{p,q} + \sum_{(p,q)\in D'} a_{m,n,p,q} x_{p,q}.\\
y_{m_\alpha,n_\beta} - \lambda = {} & \sum_{(p,q)\in(D_1\cup D_2)\cap D'} a_{m_\alpha,n_\beta,p,q} x_{p,q} + \sum_{(p,q)\in((\mathbb{N}\times\mathbb{N})\backslash(D_1\cup D_2))\cap D'} a_{m_\alpha,n_\beta,p,q} x_{p,q}\\
& + \sum_{(p,q)\in D} a_{m_\alpha,n_\beta,p,q} x_{p,q} - \lambda\\
= {} & \sum_{(p,q)\in(D_1\cup D_2)\cap D'} a_{m_\alpha,n_\beta,p,q} x_{p,q} + \sum_{(p,q)\in((\mathbb{N}\times\mathbb{N})\backslash(D_1\cup D_2))\cap D'} a_{m_\alpha,n_\beta,p,q} (\epsilon_{p,q} - \lambda)\\
& + \sum_{(p,q)\in D} a_{m_\alpha,n_\beta,p,q} x_{p,q} - \lambda\\
= {} & \sum_{(p,q)\in(D_1\cup D_2)\cap D'} a_{m_\alpha,n_\beta,p,q} x_{p,q} + \lambda \left[\sum_{(p,q)\in((\mathbb{N}\times\mathbb{N})\backslash(D_1\cup D_2))\cap D'} a_{m_\alpha,n_\beta,p,q} - 1 \right]\\
& + \sum_{(p,q)\in((\mathbb{N}\times\mathbb{N})\backslash(D_1\cup D_2))\cap D'} a_{m_\alpha,n_\beta,p,q} (\epsilon_{p,q}) + \sum_{(p,q)\in D} a_{m_\alpha,n_\beta,p,q} x_{p,q}.
\end{aligned}$$

Note that RH_1, RH_3, and RH_4 imply that

$$P\text{-}\lim_{\alpha,\beta} \sum_{(p,q)\in(D_1\cup D_2)\cap D'} |a_{m_\alpha,n_\beta,p,q}| = 0. \tag{3.1}$$

Since A is RH-regular and

$$P\text{-}\lim_{p,q\in((\mathbb{N}\times\mathbb{N})\backslash(D_1\cup D_2))\cap D'} \epsilon_{p,q} = 0$$

then clearly

$$P\text{-}\lim_{\alpha,\beta} \sum_{(p,q)\in((\mathbb{N}\times\mathbb{N})\backslash(D_1\cup D_2))\cap D'} a_{m_\alpha,n_\beta,p,q} \epsilon_{p,q} = 0. \tag{3.2}$$

Condition (3) implies that

$$P\text{-}\lim_{\alpha,\beta} \sum_{(p,q)\in((\mathbb{N}\times\mathbb{N})\backslash(D_1\cup D_2))\cap D'} a_{m_\alpha,n_\beta,p,q} = 1. \tag{3.3}$$

Eq. (3.1) through (3.3) imply

$$|y_{m_\alpha,n_\beta} - \lambda| \leq \sum_{(p,q)\in D} |a_{m_\alpha,n_\beta,p,q}||x_{p,q}| + o(1).$$

Now let us consider the following sum

$$\sum_{(p,q)\in D} a_{m_\alpha,n_\beta,p,q}.$$

Since

$$P\text{-}\lim_{\alpha,\beta} \sum_{p,q=0,0}^{\infty,\infty} |a_{m_\alpha,n_\beta,p,q}| = 1$$

there exist $\alpha \geq \alpha_0$ and $\beta \geq \beta_0$ such that

$$\sum_{p,q=0,0}^{\infty,\infty} |a_{m_\alpha,n_\beta,p,q}| \leq 1 + \epsilon.$$

Since

$$\sum_{p,q=0,0}^{\infty,\infty} |a_{m_\alpha,n_\beta,p,q}| = \sum_{(p,q)\in D} |a_{m_\alpha,n_\beta,p,q}| + \sum_{(p,q)\in D'} |a_{m_\alpha,n_\beta,p,q}|$$

and (3) implies

$$P\text{-}\lim_{\alpha,\beta} \sum_{(p,q)\in D} |a_{m_\alpha,n_\beta,p,q}| = 1$$

for $\alpha \geq \alpha_1$ and $\beta \geq \beta_1$ we are granted the following:

$$\left| \sum_{(p,q)\in D} |a_{m_\alpha,n_\beta,p,q}| - 1 \right| < \epsilon.$$

In addition, note that

$$1 - \epsilon \leq \left| \sum_{(p,q)\in D} a_{m_\alpha,n_\beta,p,q} \right| \leq \sum_{(p,q)\in D} |a_{m_\alpha,n_\beta,p,q}|$$

for all $\alpha, \beta \geq \alpha_1(\epsilon), \beta_1(\epsilon)$. Thus for $\alpha \geq \max\{\alpha_0, \alpha_1\}$ and $\beta \geq \max\{\beta_1, \beta_1\}$ we are granted the following:

$$1 - \epsilon \leq \sum_{(p,q)\in D} |a_{m_\alpha,n_\beta,p,q}| \leq 1 + \epsilon.$$

Thus $P\text{-}\lim_{\alpha,\beta} \sum_{(p,q)\in D} |a_{m_\alpha,n_\beta,p,q}| = 1$ which yields

$$P\text{-}\lim_{\alpha,\beta} \sum_{(p,q)\in D'} |a_{m_\alpha,n_\beta,p,q}| = 0. \tag{3.4}$$

Thus Eqs. (3.1) through (3.4) imply $|y_{m_\alpha,n_\beta} - \lambda| \leq \epsilon$. Thus λ is a P-limit point of $[Ax]$. Therefore $\beta^2 - C\{x\} \subseteq P - C\{Ax\}$. This completes the proof. □

The following theorem is a multidimensional analogue of Orhan and Yardimci's Theorem 1 from [6].

Theorem 3.2. *Let $[x]$ be a bounded double sequence, and let A be a strongly RH-regular matrix. Then $P - C(Ax) \subseteq \beta^2 - C(x)$ if and only if A is absolutely equivalent to a nonnegative strongly RH-regular matrix B for all bounded double sequences.*

Proof. Suppose A is absolutely equivalent to a nonnegative strongly RH-regular matrix for all bounded double sequences; then by Theorem 3.1 of [5] $P - C(Bx) \subseteq \beta^2 - C(x)$. Also since A and B are absolutely equivalent then $P - C(Ax) = P - C(Bx)$ by Lemma 3.1 of [9]. Thus $P - C(Ax) \subseteq \beta^2 - C(x)$.

Let $[x]$ be a bounded double sequence and let A be a strongly RH-regular matrix with $P - C(Ax) \subseteq \beta^2 - C(x) \subseteq P - C(x)$; then by Theorem 3.1 of [9] there exists a matrix B that is absolutely equivalent to A. In addition, Proposition 3.1 of [9] grants us

$$P\text{-}\lim_{m,n} \sum_{k,l} |a_{m,n,k,l} - b_{m,n,k,l}| = 0. \tag{3.5}$$

Thus it remains only to show the following:

(1) $P\text{-}\lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} | \triangle_{10}\, a_{m,n,k,l}| = 0$ and
(2) $P\text{-}\lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} | \triangle_{01}\, a_{m,n,k,l}| = 0$.

Let us establish (1). Eq. (3.5) and strongly RH-regularity of A grant us the following:

$$\begin{aligned} \sum_{k,l=1,1}^{\infty,\infty} |b_{m,n,k,l} - b_{m,n,k,l+1}| &\leq \sum_{k,l=1,1}^{\infty,\infty} |b_{m,n,k,l} - a_{m,n,k,l}| + \sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l+1} - b_{m,n,k,l+1}| \\ &= \sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l} - a_{m,n,k,l+1}| \\ &= \sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l+1} - b_{m,n,k,l+1}| + o(1). \end{aligned}$$

Now since A is absolutely equivalent it follows that

$$P\text{-}\lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l+1} - b_{m,n,k,l+1}| = 0.$$

The method needed to establish (2) is the same method as was used in establishing (1) and is thus omitted. This completes the proof. □

References

[1] H.J. Hamilton, Transformations of multiple sequences, Duke Math. J. 2 (1936) 29–60.
[2] K. Knopp, Zur theorie der limitierungsverfahren (Erste Mitteilung), Math. Z. 31 (1930) 115–127.
[3] G.G. Lorentz, A contribution to the theory of divergent sequences, Acta Math. 80 (1948) 167–190.
[4] F. Móricz, B.E. Rhoades, Almost convergence of double sequences and strong regularity of summability matrices, Math. Proc. Cambridge Philos Soc. 104 (1988) 283–294.
[5] Mursaleen, O. Edely, H.H. , Almost convergent and a core theorem for double sequences, J. Math. Anal. Appl. 293 (2004) 532–540.
[6] C. Orhan, Ş. Yardimci, Banach and statistical cores of bounded sequences, Czechoslovak Math. J. 54 (129) (2004) 65–72.
[7] R.F. Patterson, Analogues of some fundamental theorems of summability theory, Int. J. Math. Math. Sci. 23 (9) (2000) 1–9.
[8] R.F. Patterson, Double sequence core theorems, Int. J. Math. Math. Sci. 22 (4) (1999) 785–793.
[9] R.F. Patterson, Four dimensional matrix characterization of Pringsheim core, Southeast Asian Bull. Math. 27 (2003) 1–8.
[10] A. Pringsheim, Zur theorie der zweifach unendlichen zahlenfolgen, Math. Ann. 53 (1900) 289–321.
[11] G.M. Robison, Divergent double sequences and series, Amer. Math. Soc. Transl. 28 (1926) 50–73.
[12] L.L. Silverman, On the Definition of the Sum of a Divergent Series, Thesis, University of Missouri Studies, Mathematics (unpublished).
[13] O. Toeplitz, Über Allgenmeine Linear Mittelbrildungen, in: Prace Matemalyczno Fizyczne (Warsaw), 22, 1911.
[14] S. Yardimci, Core equality theorems for real bounded sequences, Indian J. Pure Appl. Math. 31 (2) (2000) 129–135.